The Dirac particle on central backgrounds and the anti-de Sitter oscillator

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Abstract

It is shown that, for spherically symmetric static backgrounds, a simple reduced Dirac equation can be obtained by using the Cartesian tetrad gauge in Cartesian holonomic coordinates. This equation is manifestly covariant under rotations so that the spherical coordinates can be separated in terms of angular spinors like in special relativity, obtaining a pair of radial equations and a specific form of the radial scalar product. As an example, we analytically solve the anti-de Sitter oscillator giving the formula of the energy levels and the form of the corresponding eigenspinors.

1 Introduction

In the gauge field theory [1] on curved space-time the physical meaning does not depend on the choice of the holonomic (natural) frame, or on the gauge of the tetrad field which defines the local ones [2, 3]. However, from the observer's point of view, these frames are not completely equivalent since the concrete space-time behavior depends on their choice. In general, this is studied with the help of geometric models, which play here the role of

kinetics. In the last years, many quantum models, involving Klein-Gordon or Dirac test particles on given backgrounds, have been worked out with the hope to find analytic solutions. The main results are the quantum modes of the Klein-Gordon particle on anti-de Sitter static backgrounds [4] or on several deformations of them [5], as well as the solutions of the Dirac equation on space-times with particular metrics of special interest [6, 7].

An important case is that of the Dirac equation on spherically symmetric (central) static charts which have the global symmetry of the group $T(1) \otimes SO(3)$, of time translations and rotations of the Cartesian space coordinates. There is a gauge in which the tetrad field in spherical coordinates has only diagonal components and another one where the axes of the local frame, defined by the tetrad field, are parallel with those of the Cartesian natural frame. Usually these are referred as the diagonal tetrad gauge and Cartesian gauge, respectively [8]. In general, for deriving the Dirac equation one prefers the diagonal tetrad gauge where the result is obtained directly in spherical coordinates [9]. Moreover, when one use the Cartesian gauge this is written also in spherical coordinates [8, 7]. Despite of the obvious advantages of these coordinates we believe that the study of the Dirac equation in Cartesian gauge and Cartesian natural coordinates is also interesting since, in this context, the whole theory is manifestly covariant under $T(1) \otimes SO(3)$ group. Then the energy and angular momentum are conserved like in special relativity from which we can take over the method of separation of variables.

Here we present this approach for arbitrary central static metrics. It is shown that when both the natural and local frames are Cartesian, we can put the Dirac equation in a simple form by using an appropriate transformation of the spinor field [10]. It results a reduced Dirac equation in Cartesian coordinates which is manifestly covariant under rotations. Therefore, the separation of variables in spherical coordinates can be done in terms of the angular momentum eigenspinors, like in special relativity [11, 12]. We obtain the radial equations and the form of the radial scalar product in the most general case of any central static metric, generalizing thus the well-known result of Brill and Wheeler [8]. Moreover, we show that in our approach we can easily identify the radial problems with supersymmetry, which could be analytically solvable. The example we give is of the Dirac particle on a static chart of an anti-de Sitter background (i.e. the anti-de Sitter oscillator) for which we determine the quantum modes.

We start in the second section with a short review of the main notations

and formulas. In Sec.3 we define the Cartesian gauge in Cartesian natural coordinates and we obtain the reduced Dirac equation. The next section is devoted to the separation of variables in spherical coordinates which allows us to define an independent radial problem, while in Sec.5 we discuss the cases when this has supersymmetry. The hidden supersymmetry of the anti-de Sitter oscillator is pointed out in Sec.6, where we present its complete solution giving the formula of the energy levels and the form of the energy eigenspinors up to normalization factors.

2 Preliminaries

Let us consider a chart where we have introduced the natural frame of the coordinates $x^{\mu}, \mu=0,1,2,3$. We denote by $e_{\hat{\mu}}(x)$ the tetrad fields which define the local frames and by $\hat{e}^{\hat{\mu}}(x)$ that of the corresponding coframes. These have the usual orthonormalization properties

$$e_{\hat{\mu}}\cdot e_{\hat{\nu}}=\eta_{\hat{\mu}\hat{\nu}}, \quad \hat{e}^{\hat{\mu}}\cdot\hat{e}^{\hat{\nu}}=\eta^{\hat{\mu}\hat{\nu}}, \quad \hat{e}^{\hat{\mu}}\cdot e_{\hat{\nu}}=\delta^{\hat{\mu}}_{\hat{\nu}}, \tag{1}$$

where $\eta=$diag$(1,-1,-1,-1)$ is the Minkowski metric. The 1-forms of the local frames, $d\hat{x}^{\hat{\mu}}=\hat{e}^{\hat{\mu}}_{\nu}dx^{\nu}$, allow one to write the line element

$$ds^2=\eta_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}}=g_{\mu\nu}(x)dx^{\mu}dx^{\nu}, \tag{2}$$

which defines the metric tensor $g_{\mu\nu}$ of the natural frame. This raises or lowers the Greek indices (ranging from 0 to 3) while for the hat Greek ones (with the same range) we have to use the Minkowski metric, $\eta_{\hat{\mu}\hat{\nu}}$. The derivatives $\hat{\partial}_{\hat{\nu}}=e^{\mu}_{\hat{\nu}}\partial_{\mu}$ satisfy the commutation rules

$$[\hat{\partial}_{\hat{\mu}},\hat{\partial}_{\hat{\nu}}]=e^{\alpha}_{\hat{\mu}}e^{\beta}_{\hat{\nu}}(\hat{e}^{\hat{\sigma}}_{\alpha,\beta}-\hat{e}^{\hat{\sigma}}_{\beta,\alpha})\hat{\partial}_{\hat{\sigma}}=C_{\hat{\mu}\hat{\nu}\cdot}^{\cdot\cdot\hat{\sigma}}\hat{\partial}_{\hat{\sigma}} \tag{3}$$

defining the Cartan coefficients which help us to write the conecttion components in the local frames as

$$\hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\nu}}=e^{\alpha}_{\hat{\mu}}e^{\beta}_{\hat{\nu}}(\hat{e}^{\hat{\sigma}}_{\beta,\alpha}+\hat{e}^{\hat{\sigma}}_{\gamma}\Gamma^{\gamma}_{\alpha\beta})=\frac{1}{2}\eta^{\hat{\sigma}\hat{\lambda}}(C_{\hat{\mu}\hat{\nu}\hat{\lambda}}+C_{\hat{\lambda}\hat{\mu}\hat{\nu}}+C_{\hat{\lambda}\hat{\nu}\hat{\mu}}) \tag{4}$$

while the notation $\Gamma^{\gamma}_{\alpha\beta}$ stands for the usual Christoffel symbols.

Let ψ be a Dirac free field of mass M, defined on the space domain D. In natural units, $\hbar = c = 1$, its gauge invariant action [13] is

$$\mathcal{S}[\psi] = \int_D d^4x\sqrt{-g}\left\{\frac{i}{2}[\bar{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi - (\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi] - M\bar{\psi}\psi\right\} \tag{5}$$

where

$$D_{\hat{\alpha}} = \hat{\partial}_{\hat{\alpha}} + \frac{i}{2}S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}}\hat{\Gamma}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}} \tag{6}$$

are the covariant derivatives of the spinor field and $g = \det(g_{\mu\nu})$. The Dirac matrices, $\gamma^{\hat{\alpha}}$, and the generators of the reducible spinor representation of the $SL(2,C)$ group, $S^{\hat{\alpha}\hat{\beta}}$, satisfy

$$\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}, \qquad [\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}] = -4iS^{\hat{\alpha}\hat{\beta}}, \tag{7}$$

$$[S^{\hat{\alpha}\hat{\beta}}, \gamma^{\hat{\mu}}] = i(\eta^{\hat{\beta}\hat{\mu}}\gamma^{\hat{\alpha}} - \eta^{\hat{\alpha}\hat{\mu}}\gamma^{\hat{\beta}}). \tag{8}$$

Thereby it results that the field equation,

$$i\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi - M\psi = 0, \tag{9}$$

derived from (5) can be written as

$$i\gamma^{\hat{\alpha}}e^{\mu}_{\hat{\alpha}}\partial_\mu\psi - M\psi + \frac{i}{2}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}e^{\mu}_{\hat{\alpha}})\gamma^{\hat{\alpha}}\psi - \frac{1}{4}\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}}\}\hat{\Gamma}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}\psi = 0. \tag{10}$$

On the other hand, from the conservation of the electric charge, we can deduce that when $e^0_i = 0$, $i = 1, 2, 3$, then the time-independent relativistic scalar product of two spinors is [13]

$$(\psi, \psi') = \int_D d^3x\mu(x)\bar{\psi}(x)\gamma^0\psi'(x), \tag{11}$$

where

$$\mu(x) = \sqrt{-g(x)}e^0_0(x) \tag{12}$$

is the specific weight function of the Dirac field.

3 The reduced Dirac equation

Our aim is to discuss here only the case of the charts with the global symmetry of the $T(1) \otimes SO(3)$ group. These have natural frames of the Cartesian coordinates $x^0 = t$ and x^i, $i = 1, 2, 3$, in which the metric tensor is time-independent and manifestly covariant under the rotations $R \in SO(3)$ of the space coordinates,

$$x^\mu \to x'^\mu = (Rx)^\mu \qquad (t' = t, \quad x'^i = R_{ij}x^j). \tag{13}$$

The most general form of a such a metric is given by the line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = A(r)dt^2 - [B(r)\delta_{ij} + C(r)x^i x^j]dx^i dx^j \tag{14}$$

where A, B and C are arbitrary functions of the Euclidian norm of $\vec{x}$, $r = |\vec{x}|$ (which is invariant under rotations). In applications it is convenient to replace these functions by new ones, u, v and w, such that

$$A = w^2, \quad B = \frac{w^2}{v^2}, \quad C = \frac{w^2}{r^2}\left(\frac{1}{u^2} - \frac{1}{v^2}\right). \tag{15}$$

Then the metric appears as the conformal transformation of that simpler one having $w = 1$.

Starting with a Cartesian natural frame we define the Cartesian gauge in which the static tetrad field transforms under the rotations (13) according to the rule

$$d\hat{x}^{\hat{\mu}} \to d\hat{x}'^{\hat{\mu}} = \hat{e}^{\hat{\mu}}_{\alpha}(x')dx'^{\alpha} = (Rd\hat{x})^{\hat{\mu}}. \tag{16}$$

In the case of the metric (14) the simplest choice of their components is

$$\hat{e}^0_0 = \hat{a}(r), \quad \hat{e}^0_i = \hat{e}^i_0 = 0, \quad \hat{e}^i_j = \hat{b}(r)\delta_{ij} + \hat{c}(r)x^i x^j, \tag{17}$$

$$e^0_0 = a(r), \quad e^0_i = e^i_0 = 0, \quad e^i_j = b(r)\delta_{ij} + c(r)x^i x^j, \tag{18}$$

where, according to (2), (14) and (15), we must have

$$\hat{a} = w, \quad \hat{b} = \frac{w}{v}, \quad \hat{c} = \frac{1}{r^2}\left(\frac{w}{u} - \frac{w}{v}\right), \tag{19}$$

$$a = \frac{1}{w}, \quad b = \frac{v}{w}, \quad c = \frac{1}{r^2}\left(\frac{u}{w} - \frac{v}{w}\right), \tag{20}$$

while the weight function (12) becomes

$$\mu = \frac{1}{b^2(b + r^2c)} = \frac{w^3}{uv^2} \tag{21}$$

since

$$\sqrt{-g} = B[A(B + r^2C)]^{1/2} = \frac{1}{ab^2(b + r^2c)} = \frac{w^4}{uv^2}. \tag{22}$$

From (19) and (20) we see that the function w must be positively defined in order to keep the same sense for the time axes of the natural and local frames. In addition, it is convenient to consider that the function u is positively defined too. However, the function v has an arbitrary sign. It can be represented as

$$v = \eta_P |v| \tag{23}$$

where η_P gives the relative parity. More precisely, when $\eta_P = 1$ then the space axes of the local frame are parallel with those of the natural frame, while if $\eta_P = -1$ these are antiparalel.

Now we have to replace the concrete form of the tetrad components in Eq.(10). First we eliminate its last term since it is known that this can not contribute when the metric is spherically symmetric. The argument is that $\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\hat{\gamma}}\} = \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\cdot}{}_{\hat{\lambda}}\gamma^5\gamma^{\hat{\lambda}}$ (with $\varepsilon^{0123} = 1$) is completely antisymmetric, while the Cartan coefficients resulted from (17) and (18) have no such kind of components. Furthermore, in order to simplify the remaining equation, we introduce the *reduced* Dirac field, $\tilde{\psi}$, defined by

$$\psi(x) = \chi(r)\tilde{\psi}(x). \tag{24}$$

where

$$\chi = [\sqrt{-g}(b + r^2c)]^{-1/2} = b\sqrt{a} = vw^{-3/2}. \tag{25}$$

After this transformation we obtain the reduced Dirac equation in Cartesian coordinates and Cartesian tetrad gauge,

$$i\{a(r)\gamma^0\partial_t + b(r)(\vec{\gamma}\cdot\vec{\partial}) + c(r)(\vec{\gamma}\cdot\vec{x})[1 + (\vec{x}\cdot\vec{\partial})]\}\tilde{\psi}(x) - M\tilde{\psi}(x) = 0. \tag{26}$$

This is expressed only in terms of familiar three-dimensional scalar products and scalar functions so that it is manifestly covariant under rotations. Consequently, all the properties related to the conservation of the angular momentum, including the separation of variables in spherical coordinates, will be

similar as those of the usual Dirac theory in the Minkowski flat space-time. Moreover, we can verify that here the discrete transformations, P, C and T, have the same significance as those of special relativity [11, 12]. Thus, for example, the charge conjugation transforms each particular solution of positive frequency of (26) into the corresponding one of negative frequency.

4 The radial problem

The next step is to introduce the spherical coordinates, r, θ, ϕ, associated with the space coordinates of our natural Cartesian frame. Then from (14) and (15) we obtain the line element

$$ds^2 = w^2 \left[dt^2 - \frac{dr^2}{u^2} - \frac{r^2}{v^2}(d\theta^2 + \sin^2\theta d\phi^2) \right] . \tag{27}$$

Since this is static the energy, E, is conserved. On the other hand, the form of the reduced Dirac equation (26) allows us to separate the spherical variables as in the case of the central motion in flat space-time, by using the four-components angular spinors $\Phi^{\pm}_{m_j,\kappa_j}(\theta,\phi)$ as given in Ref.[12]. These are orthogonal to each other being completely determined by the quantum number, j, of the angular momentum, the quantum number, m_j, of its projection along the third axis (of the Cartesian frame) and the value of $\kappa_j = \pm(j+1/2)$. Based on these arguments, we consider the particular solution of positive frequency

$$\tilde{\psi}_{E,j,m_j,\kappa_j}(t,r,\theta,\phi) = \frac{1}{r}[f^{(+)}(r)\Phi^{+}_{m_j,\kappa_j}(\theta,\phi) + f^{(-)}(r)\Phi^{-}_{m_j,\kappa_j}(\theta,\phi)]e^{-iEt}. \tag{28}$$

By replacing it in (26), after a little calculation, one finds that the radial functions $f^{(\pm)}$ must satisfy the radial equations

$$\left[u(r)\frac{d}{dr} + v(r)\frac{\kappa_j}{r} \right] f^{(+)}(r) = [E + w(r)M]f^{(-)}(r), \tag{29}$$

$$\left[-u(r)\frac{d}{dr} + v(r)\frac{\kappa_j}{r} \right] f^{(-)}(r) = [E - w(r)M]f^{(+)}(r). \tag{30}$$

In practice, these can be written directly starting with the line element put in the form (27) from which we have to identify the functions u, v, and w we need.

The angular spinors are normalized such that the angular integral of the scalar product (11) does not contribute and, consequently, this reduces to the radial integral. By using (24) and (28) we find that this is

$$(\tilde{\psi}_1, \tilde{\psi}_2) = \int_{D_r} \frac{dr}{u(r)} \{[f_1^{(+)}(r)]^* f_2^{(+)}(r) + [f_1^{(-)}(r)]^* f_2^{(-)}(r)\} \tag{31}$$

where D_r is the radial domain corresponding to D. What is remarkable here is that the weight function $\mu\chi^2 = 1/u$, resulted from (21) and (25), is just that we need in order to have $(u\partial_r)^+ = -u\partial_r$. This means that the operators of the left-hand side of the radial equations, are related between them through the Hermitian conjugation with respect to the scalar product (31).

A direct consequence is that the operator

$$H = \left| \begin{array}{cc} Mw & -u\dfrac{d}{dr} + \kappa_j \dfrac{v}{r} \\ u\dfrac{d}{dr} + \kappa_j \dfrac{v}{r} & -Mw \end{array} \right| \tag{32}$$

is self-adjoint. This is the radial Hamiltonian, which allows one to write the Eqs.(29) and (30) as the eigenvalue problem

$$H\mathcal{F} = E\mathcal{F}, \tag{33}$$

where the two-dimensional eigenvectors, $\mathcal{F} = |f^{(+)}, f^{(-)}|^T$, have their own scalar product,

$$(\mathcal{F}_1, \mathcal{F}_2) = \int_{D_r} \frac{dr}{u} \mathcal{F}_1^+ \mathcal{F}_2, \tag{34}$$

as it results from (31). Thus we have obtained an independent radial problem which must be solved in each particular case separately by using appropriate methods.

5 Supersymmetry in special frames

First of all, we look for possible transformations which should simplify the radial equations. It is known that the transformations of the space coordinates of a natural frame with static metric do not change the quantum modes. The simplest ones are the changes of the radial coordinate which allow us to choose suitable frames with spherical symmetry. In our opinion, the best choice is that in which the radial coordinate is defined by

$$r_s(r) = \int \frac{dr}{u(r)} + const \tag{35}$$

so that $r_s(0) = 0$. This will be called the radial coordinate of the *special* frame. In the following we shall use only this frame by taking directly $u = 1$ while the subscript s will be omitted.

Other transformations which could simplify the radial problem are the unitary transformations, $\mathcal{F} \to \hat{\mathcal{F}} = U\mathcal{F}$ and $H \to \hat{H} = UHU^+$. We shall take only those unitary matrices, U, which commute with the term of H containing derivatives. It is clear that these are nothing else than simple rotations of the plane $\{f^{(+)}, f^{(-)}\}$. Furthermore, like in the Dirac theory in flat space-time [12], we shall say that a radial problem has supersymmetry if there exist a rotation of this kind such that the transformed Hamiltonian takes the form

$$\hat{H} = \left| \begin{array}{cc} \nu & -\frac{d}{dr} + W \\ \frac{d}{dr} + W & -\nu \end{array} \right| \tag{36}$$

where ν must be a constant and W is the resulting superpotential [14]. If the radial problem has this property, then the second order equations for the components $\hat{f}^{(+)}$ and $\hat{f}^{(-)}$ of $\hat{\mathcal{F}}$ can be obtained from $\hat{H}^2\hat{\mathcal{F}} = E^2\hat{\mathcal{F}}$. These equations,

$$\left(-\frac{d^2}{dr^2} + W(r)^2 \mp \frac{dW(r)}{dr} + \nu^2 \right) \hat{f}^{(\pm)}(r) = E^2 \hat{f}^{(\pm)}(r), \tag{37}$$

represent the starting point for finding analytical solutions.

The simplest radial problems are those with manifest supersymmetry, for which the original radial Hamiltonian H has the form (36). These are generated by the metrics of the central manifolds $R \times M_3$ which have $w = 1$. In the special frames these are determined only by the arbitrary function v which gives the superpotential $W = \kappa_j v/r$.

A more complicated situation is when the metrics are conformal transformations trough w^2 of the previous ones, with functions w of the form

$$w = c_1 + c_2 \frac{v}{r} \tag{38}$$

where c_1 and c_2 are constants. In this case we need to use a suitable rotation U in order to point out the supersymmetry. These are problems with hidden supersymmetry which are similar with that of the Dirac particle in external Coulomb field, known from special relativity.

However, an example in which the supersymmetry is much more hidden will be discussed in the next section.

6 The anti-de Sitter oscillator

Let us consider a Dirac test particle on an anti-de Sitter background, in the static chart of coordinates $(t, \hat{r}, \theta, \phi)$ where the metric is given by the line element

$$ds^2 = (1 + \omega^2 \hat{r}^2)dt^2 - \frac{d\hat{r}^2}{1 + \omega^2 \hat{r}^2} - \hat{r}^2(d\theta^2 + \sin^2\theta\ d\phi^2). \tag{39}$$

In the special frame, (t, r, θ, ϕ), defined according to (35), the radial coordinate is

$$r = \frac{1}{\omega} \arctan \omega \hat{r} \tag{40}$$

and the line element becomes

$$ds^2 = \csc^2 \omega r \left[dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r\ (d\theta^2 + \sin^2\theta\ d\phi^2) \right]. \tag{41}$$

It defines a metric which covers the radial domain $D_r = [0, r_e)$, bounded by the event horizon which is at $r_e = \pi/2\omega$.

Working in the special frame we have $u = 1$. The other two functions can be identified from (41) as

$$w(r) = \csc \omega r, \quad v(r) = \omega r \sec \omega r. \tag{42}$$

With their help and by using the notation $k = M/\omega$ (i.e. $Mc^2/\hbar\omega$ in usual units), we obtain the Hamiltonian of the radial problem

$$H = \left| \begin{array}{cc} \omega k \csc \omega r & -\frac{d}{dr} + \omega\kappa_j \sec \omega r \\ \frac{d}{dr} + \omega\kappa_j \sec \omega r & -\omega k \csc \omega r \end{array} \right| . \tag{43}$$

Its form suggests us to transform $\mathcal{F}$ into $\hat{\mathcal{F}} = U(r)\mathcal{F}$ by using the local rotation

$$U(r) = \left| \begin{array}{cc} \cos \frac{\omega r}{2} & -\sin \frac{\omega r}{2} \\ \sin \frac{\omega r}{2} & \cos \frac{\omega r}{2} \end{array} \right| . \tag{44}$$

A little calculation shows us that the transformed Hamiltonian,

$$\hat{H} = U(r) H U^{+}(r) - \frac{\omega}{2} 1_{2\times 2}, \tag{45}$$

which gives the eigenvalue problem

$$\hat{H}\hat{\mathcal{F}} = \left(E - \frac{\omega}{2} \right) \hat{\mathcal{F}}, \tag{46}$$

has supersymmetry since it has the requested specific form (36) with $\nu = \omega(k - \kappa_j)$ and the superpotential

$$W(r) = \omega(k \tan \omega r + \kappa_j \cot \omega r). \tag{47}$$

Consequently, the components $\hat{f}^{(\pm)}$ of $\hat{\mathcal{F}}$ satisfy the second order equations

$$\left(-\frac{d^2}{dr^2} + \omega^2 \frac{k(k \mp 1)}{\cos^2 \omega r} + \omega^2 \frac{\kappa_j(\kappa_j \pm 1)}{\sin^2 \omega r} \right) \hat{f}^{(\pm)}(r) = \omega^2 \epsilon^2 \hat{f}^{(\pm)}(r), \tag{48}$$

where $\epsilon = E/\omega - 1/2$.

These equations are well-studied. Their solutions can be expressed in terms of hypergeometric functions [15] depending on the new variable $y = \sin^2 \omega r$ as

$$\hat{f}^{(\pm)}(y) = N_{\pm}(1-y)^{p_{\pm}} y^{s_{\pm}} F\left(s_{\pm} + p_{\pm} - \frac{\epsilon}{2}, s_{\pm} + p_{\pm} + \frac{\epsilon}{2}, 2s_{\pm} + \frac{1}{2}, y \right). \tag{49}$$

They are considered on the domain $D_y = [0, 1)$ corresponding to D_r and depend on the parameters $p_{\pm}$ and $s_{\pm}$ which must accomplish

$$2s_{\pm}(2s_{\pm} - 1) = \kappa_j(\kappa_j \pm 1) \tag{50}$$

$$2p_{\pm}(2p_{\pm} - 1) = k(k \mp 1). \tag{51}$$

Thus we have the general form of the solutions of the second order equations up to normalization factors, $N_\pm$. It remains to precise the values of the parameters $s_\pm$ and $p_\pm$ and the value of N_+/N_- so that the functions $\hat{f}^{(\pm)}$ should be solutions of the transformed radial problem (46), with a good physical meaning.

The hypergeometric functions on the domain $[0, 1)$ can be either polynomials or analytical functions strongly divergent for $y \to 1$ which can not be interpreted as tempered distributions corresponding to continuous energy levels. Therefore, we have to look only for square integrable eigenfunctions of the discrete energy spectrum. These can be obtained by choosing regular solutions in $y = 0$ and $y = 1$, with $p_\pm \geq 0$ and $s_\pm \geq 0$, and by imposing the particle-like (with $\epsilon > 0$) quantization conditions

$$\epsilon = 2(n_\pm + s_\pm + p_\pm) \tag{52}$$

which must be compatible, i.e.

$$n_+ + s_+ + p_+ = n_- + s_- + p_-. \tag{53}$$

Consequently, the solutions (49) become

$$\hat{f}^{(\pm)}(y) = N_\pm(1-y)^{p_\pm} y^{s_\pm} F\left(-n_\pm, 2s_\pm + 2p_\pm + n_\pm, 2s_\pm + \frac{1}{2}, y\right) \tag{54}$$

In the following we shall select the values of the parameters involved herein for each type of solution separately by using only one radial quantum number, n_r. Moreover, it is convenient to consider explicitly the value of the orbital angular momentum quantum number, l, of the angular spinor $\Phi^+_{m_j,\kappa_j}$, as an auxiliary quantum number.

Let us take first $\kappa_j = -(j + 1/2) = -l - 1$. Then the positive solutions of the equations (50) are

$$2s_+ = l + 1, \quad 2s_- = l + 2, \quad 2p_+ = k, \quad 2p_- = k + 1, \tag{55}$$

while, according to (53), we must have

$$n_+ = n_r, \quad n_- = n_r - 1. \tag{56}$$

Furthermore, we verify that, for these values of the parameters, the functions (54) represent a solution of the transformed radial problem if and only if

$$\frac{N_-}{N_+} = -\frac{2n_r}{2l+1} \tag{57}$$

Thus we arrive at the final result in the special frame,

$$\begin{aligned}
\hat{f}^{(+)}(r) &= \left(l+\frac{1}{2}\right)\cos^k \omega r \sin^{l+1} \omega r \\
&\quad \times F\left(-n_r, n_r+k+l+1, l+\frac{3}{2}, \sin^2 \omega r\right) \\
\hat{f}^{(-)}(r) &= -n_r \cos^{k+1} \omega r \sin^{l+2} \omega r \\
&\quad \times F\left(-n_r+1, n_r+k+l+2, l+\frac{5}{2}, \sin^2 \omega r\right)
\end{aligned} \tag{58}$$

For $\kappa_j = j + 1/2 = l$ we find

$$2s_+ = l+1, \quad 2s_- = l, \quad 2p_+ = k, \quad 2p_- = k+1, \tag{59}$$

$$n_+ = n_- = n_r \tag{60}$$

and

$$\frac{N_-}{N_+} = \frac{2l+1}{2n_r+2k+1} \tag{61}$$

so that the solutions can be written as

$$\begin{aligned}
\hat{f}^{(+)}(r) &= \left(n_r+k+\frac{1}{2}\right)\cos^k \omega r \sin^{l+1} \omega r \\
&\quad \times F\left(-n_r, n_r+k+l+1, l+\frac{3}{2}, \sin^2 \omega r\right) \\
\hat{f}^{(-)}(r) &= \left(l+\frac{1}{2}\right)\cos^{k+1} \omega r \sin^{l} \omega r \\
&\quad \times F\left(-n_r, n_r+k+l+1, l+\frac{1}{2}, \sin^2 \omega r\right)
\end{aligned} \tag{62}$$

The energy levels result from (52). Bearing in mind that $\omega k = M$ and $\omega\epsilon = E - \omega/2$, and by using the main quantum number $n = 2n_r + l$ we obtain

$$E_n = M + \omega\left(n + \frac{3}{2}\right). \tag{63}$$

These levels are degenerated. For a given n our auxiliary quantum number l takes either all the odd values from 1 to n if n is odd, or the even values from 0 to n if n is even. In both cases we have $j = l \pm 1/2$ for each l which means that $j = 1/2, 3/2, ..., n+1/2$. The selection rule for κ_j is more complicated

since it is determined by both the quantum numbers n and j. If n is even then the even κ_j are positive while the odd κ_j are negative. For odd n we are in the opposite situation, with odd positive or even negative values of κ_j. Thus it is clear that for each given pair (n,j) we have only one value of κ_j. This means that the degree of degeneracy of the level E_n is $n+1$.

On the other hand, if we know the values of n, j and κ_j, then we can find those of the quantum numbers n_r and l of the solutions (58) and (62). For this reason these will be denoted by $\hat{f}^{(\pm)}_{n,j}$. With their help we can write the components of $\mathcal{F}$ by using the inverse of (44). Finally, from (28), (24) and (25) we restore the form of the positive frequency energy eigenspinors in the special frame, up to a normalization factor, $N_{n,j}$,

$$
\begin{aligned}
&u_{n,j,m_j}(r,\theta,\phi) = \\
&= N_{n,j}\sec\omega r\cos^{3/2}\omega r\left[\left(\cos\frac{\omega r}{2}\hat{f}^{(+)}_{n,j}(r)+\sin\frac{\omega r}{2}\hat{f}^{(-)}_{n,j}(r)\right)\Phi^{+}_{m_j,\kappa_j}(\theta,\phi)\right.\\
&+\left.\left(-\sin\frac{\omega r}{2}\hat{f}^{(+)}_{n,j}(r)+\cos\frac{\omega r}{2}\hat{f}^{(-)}_{n,j}(r)\right)\Phi^{-}_{m_j,\kappa_j}(\theta,\phi)\right]. \qquad (64)
\end{aligned}
$$

The negative frequency eigenspinors can be derived directly by using the charge conjugation [11]. These are

$$
v_{n,j,m_j} = (u_{n,j,m_j})^c \equiv C(\bar{u}_{n,j,m_j})^T \qquad (65)
$$

where $C = i\gamma^2\gamma^0$. Thus the problem of the quantum modes of the anti-de Sitter oscillator is completely solved.

7 Comments

The above presented example shows us that our method based on the Cartesian tetrad gauge in Cartesian coordinates has similar features with that used to solve the central motion in flat space-time. We can say that, in some sense, the pair of the natural and local frames we have choose plays the same role as the rest frames from special relativity. This allowed us to separate the spherical variables in terms of the angular spinors such that all the constants involved in the separation of variables get good physical meaning. On the other hand, the complete formulation of the radial problem

is useful in applications since it includes the radial scalar product which help us to identify the radial wave functions corresponding to the discrete or continuous energy spectra.

By using this method we have obtained the quantum modes of the anti-de Sitter oscillator. In our opinion, this is the first step to the quantum theory of the Dirac field on anti-de Sitter backgrounds. It follows to calculate the normalization factors, to derive the main properties of the spinors u_{n,j,m_j} and v_{n,j,m_j}, and to introduce suitable creation and anihilations operators. Thus we hope to obtain in near future the quantum theory of a free Dirac field (in the sense of general relativity) with countable energy spectrum.

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